Quaternionic Dolbeault complex and vanishing theorems on hyperkähler manifolds

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Abstract
Let (M, I, J, K) be a compact hyperkähler manifold, $\dim_{\mathbb{H}} M = n$, and L a non-trivial holomorphic line bundle on (M, I). Using the quaternionic Dolbeault complex, we prove the following vanishing theorem for holomorphic cohomology of L. If $c_1(L)$ lies in the closure $\hat{K}$ of the dual Kähler cone, then $H^i(L) = 0$ for $i > n$. If $c_1(L)$ lies in the opposite cone $-\hat{K}$, then $H^i(L) = 0$ for $i < n$. Finally, if $c_1(L)$ is neither in $\hat{K}$ nor in $-\hat{K}$, then $H^i(L) = 0$ for $i \neq n$.

Contents

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1 Introduction

1.1 Hypercomplex and hyperkähler manifolds

Definition 1.1: Let M be a manifold, and $I, J, K \in \operatorname{End}(TM)$ endomorphisms of the tangent bundle satisfying the quaternionic relation

$$I^2 = J^2 = K^2 = IJK = -\operatorname{Id}_{TM}.$$

The manifold (M, I, J, K) is called **hypercomplex** if the almost complex structures I, J, K are integrable. If, in addition, M is equipped with a Riemannian metric g which is Kähler with respect to I, J, K, the manifold (M, I, J, K, g) is called **hyperkähler**.

Consider the Kähler forms $\omega_I, \omega_J, \omega_K$ on M:

$$\omega_I(\cdot,\cdot) := g(\cdot, I\cdot), \quad \omega_J(\cdot,\cdot) := g(\cdot, J\cdot), \quad \omega_K(\cdot,\cdot) := g(\cdot, K\cdot).$$

An elementary linear-algebraic calculation implies that the 2-form $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is of Hodge type $(2,0)$ on (M, I). This form is clearly closed and non-degenerate, hence it is a holomorphic symplectic form.

In algebraic geometry, the word "hyperkähler" is essentially synonymous with "holomorphically symplectic", due to the following theorem, which is implied by Yau's solution of Calabi conjecture.

Theorem 1.2: Let (M, I) be a compact, Kähler, holomorphically symplectic manifold. Then there exists a unique hyperkähler metric on (M, I) with the same Kähler class.

Proof: See [Y], [Bes]. ■

Remark 1.3: The hyperkähler metric is unique, but there could be several hyperkähler structures compatible with a given hyperkähler metric on (M, I).

1.2 Bogomolov's decomposition theorem

The modern approach to Bogomolov's decomposition is based on Calabi-Yau theorem (Theorem 1.2), Berger's classification of irreducible holonomy and de Rham's splitting theorem for holonomy reduction ([Bea], [Bes]). It is worth mention that the original proof of decomposition theorem (due to F. Bogomolov, [Bo1]) was algebraic.

Theorem 1.4: Let (M, I, J, K) be a compact hyperkähler manifold. Then there exists a finite covering $\widetilde{M} \longrightarrow M$, such that $\widetilde{M}$ is decomposed, as a hyperkähler manifold, into a product

$$\widetilde{M} = M_1 \times M_2 \times \ldots M_n \times T,$$

where (M_i, I, J, K) satisfy $H^1(M_i) = 0$, $H^{2,0}(M_i, I) = \mathbb{C}$, and T is a hyperkähler torus. Moreover, M_i are uniquely determined by M and simply connected, and T is unique up to isogeny.

Proof: See [Bea], [Bes].■

Definition 1.5: Let (M, I, J, K) be a compact hyperkähler manifold which satisfies $H^1(M) = 0$, $H^{2,0}(M, I) = \mathbb{C}$. Then M is called **a irreducible hyperkähler manifold**.

Remark 1.6: Notice that Theorem 1.4 implies that irreducible hyperkähler manifolds are simly connected. In particular, they do not admit a further decomposition. This explains the term "irreducible".

1.3 Vanishing theorems on hyperkähler manifolds

Using the argument which essentially belongs to the theory of hypercomplex manifolds, we are able to prove the following algebro-geometric statements.

Theorem 1.7: Let (M, I, J, K) be a compact, irreducible hyperkähler manifold, and L a holomorphic line bundle on (M, I) with $c_1(L) \neq 0$. Denote by

$$\overline{\mathcal{K}}\check{} \subset H^{1,1}(M, I) \cap H^2(M, \mathbb{R})$$

the closure of the dual Kähler cone of (M, I) (Subsection 5.2), and let $-\overline{\mathcal{K}}\check{}$ be the opposite cone. Then one of the following holds.

(i) $c_1(L) \in \overline{\mathcal{K}}\check{}$; then $H^i(L) = 0$ for $i > \frac{\dim_{\mathbb{C}} M}{2}$.

(ii) $c_1(L) \in -\overline{\mathcal{K}}\check{}$; then $H^i(L) = 0$ for $i < \frac{\dim_{\mathbb{C}} M}{2}$.

(iii) $c_1(L)$ does not lie in $-\overline{\mathcal{K}}\check{} \cup \overline{\mathcal{K}}\check{}$; then $H^i(L) = 0$ for $i \neq \frac{\dim_{\mathbb{C}} M}{2}$.

Proof: See Theorem 5.6. ■

Theorem 1.8: Let (M, I, J, K) be a compact, irreducible hyperkähler manifold, L a holomorphic line bundle on (M, I) with $c_1(L) \neq 0$, and B an arbitrary holomorphic vector bundle on (M, I). Then there exists a sufficiently big number N_0, such that for any integer $N > N_0$ one of the following holds.

(i) $c_1(L) \in \overline{\mathcal{K}}\check{}$; then $H^i(L^N \otimes B) = 0$ for $i > \frac{\dim_{\mathbb{C}} M}{2}$.

(ii) $c_1(L) \in -\overline{\mathcal{K}}\check{}$; then $H^i(L^N \otimes B) = 0$ for $i < \frac{\dim_{\mathbb{C}} M}{2}$.

(iii) $c_1(L)$ does not lie in $-\overline{\mathcal{K}}\check{} \cup \overline{\mathcal{K}}\check{}$; then $H^i(L^N \otimes B) = 0$ for $i \neq \frac{\dim_{\mathbb{C}} M}{2}$.

Proof: This is Theorem 5.8. ■

The vanishing theorems have many interesting geometrical consequences. As an example, we give the following theorem (Section 6).

Theorem 1.9: Let (M, I, J, K) be a irreducible hyperkähler manifold, and $X \subset (M, I)$ a subvariety of dimension $\dim_{\mathbb{C}} X > \frac{1}{2} \dim_{\mathbb{C}} M$. Assume that X is a complete intersection of ample divisors. Consider a holomorphic line bundle L on (M, I) with $c_1(L)$ nef (that is, $c_1(L)$ lies in the closure of the Kähler cone of (M, I)) and $q(c_1(L), c_1(L)) = 0$, where q is the Bogomolov-Beauville-Fujiki bilinear form (Definition 4.4). Then the natural restriction map is surjective on holomorphic sections:

$$H^0(L^N) \longrightarrow H^0(L^N\big|_X) \longrightarrow 0.$$

for a sufficiently big power of L.

Proof: See Theorem 6.7. ■

1.4 Quaternionic Dolbeault complex and vanishing

In this Subsection we give a brief introduction to quaternionic Dolbeault complex. We sketch how one can use the quaternionic Dolbeault complex

to deduce the vanishing theorems for cohomology. Further on in this paper, this theme is developed in a more detailed way.

Let M be a hypercomplex manifold. There is a natural action of $SU(2)$ on $\Lambda^1(M)$ (we identify $SU(2)$ with the group of unitary quaternions). This action is extended to $\Lambda^*(M)$ by multiplicativity.

This $SU(2)$-action plays the same role in hypercomplex and hyperkähler geometry as the usual Hodge decomposition in complex geometry.

Let

$$\Lambda^i(M) = \bigoplus \Lambda^i_k(M)$$

be a weight decomposition of the space of i-forms, with $\Lambda^i_k(M)$ an $SU(2)$-representation of weight k (see Subsection 2.1). It is easy to check that

$$V^* := \bigoplus_{i>k} \Lambda^i_k(M)$$

is a differential ideal in the de Rham algebra $\Lambda^*(M)$, that is, an ideal which satisfies $dV^* \subset V^*$ (Subsection 2.2). Therefore, the quotient $\Lambda^*(M)/V^*$ is a differential graded algebra, denoted as $(\Lambda^*_+(M), d_+)$. This algebra is called **the quaternionic Dolbeault complex** (Definition 2.3). We approach $(\Lambda^*_+(M), d_+)$ from the same point of view as one approaches the usual Dolbeault complex in algebraic geometry. There is a Hodge decomposition (Subsection 2.3), and a Lefschetz-type $\mathfrak{sl}(2)$-action (Proposition 3.1). The analogue of Kodaira-Nakano formula is written in (3.4):

$$\Delta_{\overline{\partial}} = \Delta_{\overline{\partial}_J} + [\Lambda_{\overline{\Omega}}, \Theta_+], \tag{1.1}$$

where $\Delta_{\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ is the usual Laplacian on $(0,p)$-forms with coefficients in a holomorphic vector bundle B on (M, I), Θ_+ the $\Lambda^2_+(M) \otimes \operatorname{End}(B)$-part of the curvature of B, and $\Delta_{\overline{\partial}_J}$ a positive self-dual operator. When the commutator $[\Lambda_{\overline{\Omega}}, \Theta_+]$ is positive, this leads to the vanishing theorems Theorem 1.7 and Theorem 1.8, which are deduced from (1.1) in the same way as Kodaira-Nakano vanishing is deduced from the Kodaira-Nakano identity.

If the bundle B is a line bundle, we can choose its metric in such a way that its curvature 2-form Θ_B is harmonic ([GH]). Consider the weight decomposition

$$\Theta_B = \Theta_+ + \Theta_0,$$

where Θ_0 is $SU(2)$-invariant. Then Θ_+ is harmonic (see Subsection 4.1). From (5.1), it follows that $\Theta_+ = \lambda\omega_I$, where λ is a real constant, and ω_I is the Kähler form of (M, I). Then

$$[\Lambda_{\overline{\Omega}}, \Theta_+] = \lambda H_{\overline{\Omega}}, \tag{1.2}$$

where H is a scalar operator mapping a $(0,p)$-form η into $(n-p)\eta$, where $n = \dim_{\mathbb{H}} M$ (see (5.4)). For $\lambda > 0$, (1.2) is positive when $p < n$, and for $\lambda < 0$, (1.2) is positive when $p > n$. The vanishing of holomorphic cohomology (for $p > n$ in the first case, and for $p < n$ in the second case) is a consequence.

2 Quaternionic Dolbeault complex

2.1 Weights of $SU(2)$-representations

It is well-known that any irreducible representations of $SU(2)$ over $\mathbb{C}$ can be obtained as a symmetric power $S^i(V_2)$, where V_1 is a fundamental 2-dimensional representation. We say that a representation W **has weight** i if it is isomorphic to $S^i(V_1)$. A representation is said to be **pure of weight** i if all its irreducible components have weight i. If all irreducible components of a representation W_1 have weight $\leqslant i$, we say that W_1 **is a representation of weight** $\leqslant i$. In a similar fashion one defines representations of weight $\geqslant i$.

Remark 2.1: The Clebsch-Gordan formula (see [H]) claims that the weight is *multiplicative*, in the following sense: if $i \leqslant j$, then

$$V_i \otimes V_j = \bigoplus_{k=0}^{i} V_{i+j-2k},$$

where $V_i = S^i(V_1)$ denotes the irreducible representation of weight i.

A subspace $W \subset W_1$ is **pure of weight** i if the $SU(2)$-representation $W' \subset W_1$ generated by W is pure of weight i.

2.2 Quaternionic Dolbeault complex: a definition

Let M be a hypercomplex (e.g. a hyperkähler) manifold, $\dim_{\mathbb{H}} M = n$. There is a natural multiplicative action of $SU(2) \subset \mathbb{H}^*$ on $\Lambda^*(M)$, associated with the hypercomplex structure.

Remark 2.2: The space $\Lambda^*(M)$ is an infinite-dimensional representation of $SU(2)$, however, all its irreducible components are finite-dimensional. Therefore it makes sense to speak of *weight* of $\Lambda^*(M)$ and its sub-representations. Clearly, $\Lambda^1(M)$ has weight 1. From Clebsch-Gordan formula

(Remark 2.1), it follows that $\Lambda^i(M)$ is an $SU(2)$-representation of weight $\leqslant i$. Using the Hodge $*$-isomorphism $\Lambda^i(M) \cong \Lambda^{4n-i}(M)$, we find that for $i > 2n$, $\Lambda^i(M)$ is a representation of weight $\leqslant 2n - i$.

Let $V^i \subset \Lambda^i(M)$ be a maximal $SU(2)$-invariant subspace of weight $< i$. The space V^i is well defined, because it is a sum of all irreducible representations $W \subset \Lambda^i(M)$ of weight $< i$. Since the weight is multiplicative (Remark 2.1), $V^* = \bigoplus_i V^i$ is an ideal in $\Lambda^*(M)$. We also have $V^i = \Lambda^i(M)$ for $i > 2n$ (Remark 2.2).

It is easy to see that the de Rham differential d increases the weight by 1 at most. Therefore, $dV^i \subset V^{i+1}$, and $V^* \subset \Lambda^*(M)$ is a differential ideal in the de Rham DG-algebra $(\Lambda^*(M), d)$.

Definition 2.3: Denote by $(\Lambda^*_+(M), d_+)$ the quotient algebra $\Lambda^*(M)/V^*$ It is called **the quaternionic Dolbeault algebra of** M, or **the quaternionic Dolbeault complex** (qD-algebra or qD-complex for short).

The space $\Lambda^i_+(M)$ can be identified with the maximal subspace of $\Lambda^i(M)$ of weight i, that is, a sum of all irreducible sub-representations of weight i. This way, $\Lambda^i_+(M)$ can be considered as a subspace in $\Lambda^i(M)$; however, this subspace is not preserved by the multiplicative structure and the differential.

Remark 2.4: The complex $(\Lambda^*_+(M), d_+)$ was constructed much earlier by Salamon, in a different (and much more general) situation, and much studied since then ([Sal], [CS], [Bas], [L]).

2.3 The Hodge decomposition on the quaternionic Dolbeault complex

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Let (M, I, J, K) be a hypercomplex manifold, and L a complex structure induced by the quaternionic action, say, I, J or K. Consider the $U(1)$-action on $\Lambda^1(M)$ provided by $\varphi \xrightarrow{\rho_L} \cos\varphi \operatorname{Id} + \sin\varphi \cdot L$. We extend this action to a multiplicative action on $\Lambda^*(M)$. Clearly, for a (p,q)-form $\eta \in \Lambda^{p,q}(M, L)$, we have

$$\rho_L(\varphi)\eta = e^{\sqrt{-1}\,(p-q)\varphi}\eta. \tag{2.1}$$

Lemma 2.5: Let (M, I, J, K) be a hypercomplex manifold and

$$\rho_I, \rho_J, \rho_K$$

the homomorphisms
$$U(1) \longrightarrow \operatorname{Aut}(\Lambda^*(M))$$
constructied above. Then ρ_I, ρ_J, ρ_K generate the Lie group action
$$SU(2) \subset \operatorname{Aut}(\Lambda^*(M))$$
associated with the hypercomplex structure.

Proof: Lemma 2.5 is clear. Indeed, the action of $SU(2)$, and ρ_I, ρ_J, ρ_K are defined on $\Lambda^*(M)$ by multiplicativity, hence it suffices to check that ρ_I, ρ_J, ρ_K generate the standard action of $SU(2)$ on $\Lambda^1(M)$. On $\Lambda^1(M)$, ρ_I, ρ_J, ρ_K act as quaternions $\cos\varphi + \sin\varphi \cdot I$, $\cos\varphi + \sin\varphi \cdot J$, $\cos\varphi + \sin\varphi \cdot K$, and they generate the group of unitary quaternions. ■

From Lemma 2.5, it is clear that ρ_L preserves components of weight i. We obtain that V^* is preserved by ρ_L, hence ρ_L acts on $\Lambda^*_+(M)$. Then, (2.1) gives a Hodge decomposition on $\Lambda^*_+(M)$:
$$\Lambda^i_+(M) = \bigoplus_{p+q=i} \Lambda^{p,q}_{+,L}(M).$$

The following result is implied immediately by the standard calculations from the theory of $SU(2)$-representations.

Proposition 2.6: Let (M, I, J, K) be a hypercomplex manifold and
$$\Lambda^i_+(M) = \bigoplus_{p+q=i} \Lambda^{p,q}_{+,I}(M)$$
the Hodge decomposition of qD-complex defined above. Then there is a natural isomorphism
$$\Lambda^{p,q}_{+,I}(M) \cong \Lambda^{0,p+q}(M, I). \tag{2.2}$$

Proof: The following lemma is clear.

Lemma 2.7: Let (M, I, J, K) be a hypercomplex manifold, $\dim_{\mathbb{H}} M = n$, and p an integer, $0 \leqslant p \leqslant 2n$. Then $\Lambda^{0,p}(M, I) \subset \Lambda^p(M)$ is pure of weight p.

Proof: Consider the operator $W_I: \Lambda^*(M) \longrightarrow \Lambda^*(M)$ mapping a form $\eta \in \Lambda^{p,q}(M, I)$ to $\sqrt{-1}\,(p-q)\eta$. Clearly, W_I acts as a generator of $\mathfrak{u}(1)$,

with $\mathfrak{u}(1)$ associated to $\rho_I : U(1) \longrightarrow \operatorname{End}(\Lambda^*(M))$. By Lemma 2.5, $W_I \in \mathfrak{su}(2)$, where the $\mathfrak{su}(2)$-action on $\Lambda^*(M)$ is associated with the standard action of $SU(2)$. Writing $\mathfrak{su}(2)$ explicitly in terms of generators W_I, W_J and W_K, we find that W_I generates a Cartan subalgebra of $\mathfrak{su}(2)$ (indeed, the corresponding Lie group is a maximal compact torus of $SU(2)$). Since the Cartan algebra $\mathbb{C}\cdot W_I$ acts on $\Lambda^{p,0}(M,I)$ with weight p, the space $\Lambda^{p,0}(M,I)$ is of weight $\geqslant p$. On the other hand, $\Lambda^p(M)$ is a representation of weight $\leqslant p$ (Remark 2.2). Therefore, $\Lambda^{p,0}(M,I)$ is pure of weight p. ∎

Remark 2.8: This argument also implies that $\Lambda^{0,p}(M,I)$ coinsides with $\Lambda^{0,p}_{+,I}(M) \subset \Lambda^p_+(M)$ (here we consider $\Lambda^p_+(M)$ as a maximal $SU(2)$-invariant subspace of weight p in $\Lambda^p(M)$).

Now, Proposition 2.6 is implied by the general machinery of $SU(2)$-representations. If R is a finite-dimensional $SU(2)$-representation of weight $\leqslant p$, the Cartan algebra action splits R onto weight components $R = \oplus R_i$, $i = -p, -p+2, \ldots\ p-2, p$ the weights of the root $\sqrt{-1}\, W_I$ acting on R_i as a multiplication by i. Moreover, if R is pure of weight p, then all spaces R_i are naturally isomorphic, with isomorphism provided by the $SU(2)$-action.

In the case $R = \Lambda^p_+(M)$, the decomposition $R = \oplus R_i$ is precisely the Hodge decomposition, hence the spaces $\Lambda^{p,q}_{+,I}(M)$ are naturally isomorphic to for all $p, q \geqslant 0$ satisfying $p + q = i$. We proved Proposition 2.6. ∎

2.4 The Hodge decomposition on qD-complex: an explicit construction

The isomorphism (2.2) can be made explicit, and also multiplicative, in the following way. Let $\mathfrak{R}$ be an irreducible 2-dimensional representation of $SU(2)$. Clearly, any irreducible $SU(2)$-representation of weight p is isomorphic to $S^p\mathfrak{R}$ (the p-th symmetric power of $\mathfrak{R}$). Consider the root $\sqrt{-1}\, W_I \in \mathfrak{su}(2)$, constructed in Subsection 2.3. The corresponding $\mathfrak{sl}(2)$-triple can be written as

$$f = W_J + \sqrt{-1}\, W_K, \quad g = W_J - \sqrt{-1}\, W_K, \quad h = \sqrt{-1}\, W_I.$$

Let x, y be a basis in $\mathfrak{R}$, such that $hx = x$, $hy = -y$, $gx = y$, $fy = x$.

Consider a hypercomplex manifold (M, I, J, K). The bundle

$$\mathfrak{S} := \bigoplus_p S^p\mathfrak{R} \otimes \Lambda^{0,p}(M,I), \tag{2.3}$$

is equipped with a natural multiplicative structure (we assume that the elements of $S^p\mathfrak{R}$ and $\Lambda^{0,q}(M,I)$ commute). We define the following $SU(2)$-action on $\mathfrak{S}$: $SU(2)$ acts trivially on $\Lambda^{0,p}(M,I)$, and in a standard way on $S^p\mathfrak{R}$.

Consider an isomorphism $\mathfrak{R}\otimes\Lambda^{0,1}(M,I)\longrightarrow\Lambda^1(M)$ mapping $x\otimes\eta$ to $J(\eta)$ and $y\otimes\eta$ to η. This map is clearly $SU(2)$-invariant. Using the multiplicative structure on $\mathfrak{S}$, it can be extended to an $SU(2)$-invariant algebra homomorphism

$$\bigoplus_p S^p\mathfrak{R}\otimes\Lambda^{0,p}(M,I)\longrightarrow\Lambda^*_+(M). \tag{2.4}$$

Proposition 2.9: In these assumptions, (2.4) is an algebra isomorphism.

Proof: Let $\mathfrak{S}^p\subset\mathfrak{S}$ denote the grading p component. Bijectivity of the map (2.4) is checked in the same way as one proves Proposition 2.6: the Hodge components of $\mathfrak{S}^p$ are all isomorphic, because $\mathfrak{S}^p$ is a pure representation of weight p, and the same is true for $\Lambda^p_+(M)$. Therefore, it suffices to prove that the restriction of (2.4) to one Hodge component, say, $y^p\Lambda^{0,p}(M,I)$, induces an isomorphism

$$y^p\Lambda^{0,p}(M,I)\longrightarrow\Lambda^{0,p}_{+,I}(M).$$

This is implied by the equality $\Lambda^{0,p}(M,I)=\Lambda^{0,p}_{+,I}(M)$ (Remark 2.8). ∎

2.5 The $\overline{\partial}_J$-operator

Let (M,I,J,K) be a hypercomplex manifold. We extend

$$J:\ \Lambda^1(M)\longrightarrow\Lambda^1(M)$$

to $\Lambda^*(M)$ by multiplicativity. Recall that

$$J(\Lambda^{p,q}(M,I))=\Lambda^{q,p}(M,I),$$

because I and J anticommute on $\Lambda^1(M)$. Denote by

$$\overline{\partial}_J:\ \Lambda^{p,q}(M,I)\longrightarrow\Lambda^{p,q+1}(M,I)$$

the operator $J\circ\partial\circ J$, where $\partial:\ \Lambda^{p,q}(M,I)\longrightarrow\Lambda^{p+1,q}(M,I)$ is the standard Dolbeault operator on (M,I), that is, the $(1,0)$-part of the de Rham differential. Since $\partial^2=0$, we have $\overline{\partial}_J^2=0$. Since I,J,K are integrable, the

operators d, $d_I := I \circ d \circ I$, $d_J := J \circ d \circ J$, $d_K := K \circ d \circ K$ pairwise anticommute. Therefore, $\overline{\partial} = \frac{d - \sqrt{-1}\, d_I}{2}$ anticommutes with $\overline{\partial}_J = \frac{d_J - \sqrt{-1}\, d_K}{2}$. Writing the supercommutator as $\{\cdot,\cdot\}$, we express this as

$$\{\overline{\partial}_J, \overline{\partial}_J\} = 0, \quad \{\overline{\partial}_J, \overline{\partial}\} = 0. \tag{2.5}$$

2.6 The $\overline{\partial}$, $\overline{\partial}_J$-bicomplex

Consider the quaternionic Dolbeault complex $(\Lambda^*_+(M), d_+)$ constructed in Subsection 2.2. Using the Hodge decomposition, we can represent this complex as

$$\begin{tikzcd}[column sep=tiny]
 & & \Lambda^0_{+,I}(M) \arrow[dl, "d^{1,0}_{+,I}"'] \arrow[dr, "d^{0,1}_{+,I}"] & & \\
 & \Lambda^{1,0}_{+,I}(M) \arrow[dl, "d^{1,0}_{+,I}"'] \arrow[dr, "d^{0,1}_{+,I}"] & & \Lambda^{0,1}_{+,I}(M) \arrow[dl, "d^{1,0}_{+,I}"'] \arrow[dr, "d^{0,1}_{+,I}"] & \\
 \Lambda^{2,0}_{+,I}(M) & & \Lambda^{1,1}_{+,I}(M) & & \Lambda^{0,2}_{+,I}(M)
\end{tikzcd} \tag{2.6}$$

where $d^{1,0}_{+,I}$, $d^{0,1}_{+,I}$ are the Hodge components of the quaternionic Dolbeault differential d_+, taken with respect to I.

Consider a hypercomplex manifold (M, I, J, K). Let

$$\bigoplus_p S^p \mathfrak{R} \otimes \Lambda^{0,p}(M, I) \longrightarrow \Lambda^*_+(M). \tag{2.7}$$

be the isomorphism constructed in Proposition 2.9. Writing the basis x, y of $\mathfrak{R}$ as in the proof of Proposition 2.9, we may write the Hodge decomposition of (2.7) as

$$x^p y^q \Lambda^{0,p+q}(M, I) \cong \Lambda^{p,q}_{+,I}(M).$$

Theorem 2.10: Under this correspondence, $d^{0,1}_+$ corresponds to $\overline{\partial}$ and $d^{1,0}_+$ to $\overline{\partial}_J$. This way the bicomplex (2.6) becomes equivalent to the bicomplex $(\Lambda^{0,p}(M, I), \overline{\partial}, \overline{\partial}_J)$ as follows:

$$
\begin{tikzcd}[column sep=small]
 & & \Lambda^0_{+,I}(M) \arrow[dl, "d^{1,0}_{+,I}"] \arrow[dr, "d^{0,1}_{+,I}"] & & & & & & \Lambda^{0,0}_I(M) \arrow[dl, "x\overline{\partial}_J"] \arrow[dr, "y\overline{\partial}"] & & \\
 & \Lambda^{1,0}_{+,I}(M) \arrow[dl, "d^{1,0}_{+,I}"] \arrow[dr, "d^{0,1}_{+,I}"] & & \Lambda^{0,1}_{+,I}(M) \arrow[dl, "d^{1,0}_{+,I}"] \arrow[dr, "d^{0,1}_{+,I}"] & & \cong & & x\Lambda^{0,1}_I(M) \arrow[dl, "x\overline{\partial}_J"] \arrow[dr, "y\overline{\partial}"] & & y\Lambda^{0,1}_I(M) \arrow[dl, "x\overline{\partial}_J"] \arrow[dr, "y\overline{\partial}"] & \\
\Lambda^{0,2}_{+,I}(M) & & \Lambda^{1,1}_{+,I}(M) & & \Lambda^{0,2}_{+,I}(M) & & x^2\Lambda^{0,2}_I(M) & & xy\Lambda^{0,2}_I(M) & & y^2\Lambda^{0,2}_I(M)
\end{tikzcd} \tag{2.8}
$$

Proof: Consider the action of $x\overline{\partial}_J + y\overline{\partial}$ on

$$\bigoplus_p S^p\mathfrak{R} \otimes \Lambda^{0,p}(M,I) \cong \Lambda^*_+(M)$$

defined as in (2.8). To prove Theorem 2.10, we need to show that

$$x\overline{\partial}_J + y\overline{\partial} = d_+. \tag{2.9}$$

Both of these operators satisfy the Leibniz rule, hence it suffices to check (2.9) on some set of multiplicative generators of $\Lambda^*_+(M)$. On $\Lambda^0_+(M)$, the equality (2.9) is clear from the definitions:

$$x\overline{\partial}_J + y\overline{\partial}\Big|_{\Lambda^0_+(M)} = d_+\Big|_{\Lambda^0_+(M)}. \tag{2.10}$$

It is easy to check that the space $\Lambda^0(M) \oplus d\Lambda^0(M)$ generates the algebra $\Lambda^*(M)$. Therefore, $\Lambda^0_+(M) \oplus d_+\Lambda^0_+(M)$ generates $\Lambda^*_+(M)$. To prove Theorem 2.10 it remains to show that

$$x\overline{\partial}_J + y\overline{\partial}\Big|_{d_+\Lambda^0_+(M)} = d_+\Big|_{d_+\Lambda^0_+(M)}. \tag{2.11}$$

Since $d_+^2 = 0$, $d_+\Big|_{d_+\Lambda^0_+(M)} = 0$. By (2.5),

$$(x\overline{\partial}_J + y\overline{\partial})^2 = 0. \tag{2.12}$$

Using (2.10) and (2.12), we obtain

$$x\overline{\partial}_J + y\overline{\partial}\Big|_{d_+\Lambda^0_+(M)} = x\overline{\partial}_J + y\overline{\partial}\Big|_{x\overline{\partial}_J + y\overline{\partial}(\Lambda^0_+(M))} = 0.$$

Therefore,

$$x\overline{\partial}_J + y\overline{\partial}\Big|_{d_+\Lambda^0_+(M)} = d_+\Big|_{d_+\Lambda^0_+(M)} = 0.$$

This proves (2.11). Theorem 2.10 is proven. ∎

3 Kodaira identities for qD-complex

3.1 The Lefschetz-type $\mathfrak{sl}(2)$-action on $\Lambda^{0,*}(M,I)\otimes \operatorname{End}(B)$

Let (M,I,J,K) be a hyperkähler manifold, B a holomorphic Hermitian vector bundle on (M,I), and $\Lambda^{0,*}(M,I)\otimes B$ the space of $(0,p)$-forms with coefficients in B. Denote by $\overline{\Omega}\in \Lambda^{0,2}(M,I)$ the standard $(0,2)$-form $\omega_J+\sqrt{-1}\,\omega_K$ (Subsection 1.1).

Using a hyperkähler metric, we constract a natural Hermitian structure on $\Lambda^{0,*}(M,I)\otimes B$. Denote by $L_{\overline{\Omega}}:\ \Lambda^{q,p}(M,I)\longrightarrow \Lambda^{q,p+2}(M)$ the operator of exterior multiplication by $\overline{\Omega}$, and by $\Lambda_{\overline{\Omega}}:\ \Lambda^{q,p}(M,I)\otimes B\longrightarrow \Lambda^{q,p-2}(M)\otimes B$ its Hermitian adjoint. The same argument as proves the usual Lefschetz Theorem about the $\mathfrak{sl}(2)$-action (see [GH]) can be used to prove the following linear-algebraic result, which is due to A. Fujiki.

Proposition 3.1: ([F]) In the above assumptions, let

$$H_{\overline{\Omega}} := [L_{\overline{\Omega}}, \Lambda_{\overline{\Omega}}]$$

be a commutator of $L_{\overline{\Omega}}, \Lambda_{\overline{\Omega}}$. Then $H_{\overline{\Omega}}$ is a scalar operator, multiplying a (q,p)-form by $n-p$, where $n=\frac{1}{2}\dim \mathbb{H}(M)$. Moreover, $L_{\overline{\Omega}}, \Lambda_{\overline{\Omega}}$, $H_{\overline{\Omega}}$ is an $\mathfrak{sl}(2)$-triple.

Proof: See [V1], Theorem 4.2). ■

Let $\theta\in\Lambda^{0,1}(M,I)\otimes \operatorname{End}(B)$ be a 1-form. Denote by

$$L_\theta:\ \Lambda^{q,p}(M,I)\otimes B\longrightarrow \Lambda^{q,p+1}(M,I)\otimes B$$

the operator of multiplication by θ, and let

$$\Lambda_\theta:\ \Lambda^{q,p}(M,I)\longrightarrow \Lambda^{q,p-1}(M,I)$$

be its Hermitian adjoint. Denote by θ_J the $(0,1)$-form $J(\overline{\theta})$.

Claim 3.2: In the above assumptions, we have

$$[L_{\overline{\Omega}}, \Lambda_\theta] = L_{\theta_J}. \qquad (3.1)$$

Proof: Follows from a trivial computation. ■

3.2 $\overline{\partial}$, $\overline{\partial}_J$ with coefficients in a bundle

Let (M, I, J, K) be a hyperkähler manifold, and B a holomorphic Hermitian vector bundle on (M, I). Consider the standard (Chern) Hermitian connection ∇ on B, $\nabla = \nabla^{1,0} + \overline{\partial}$, where $\overline{\partial}:\ B \longrightarrow B \otimes \Lambda^{0,1}(M,I)$ is the holomorphic structure operator. Denote by $\overline{\partial}_J:\ B \longrightarrow B \otimes \Lambda^{0,1}(M,I)$ the composition of $\nabla^{1,0}:\ B \otimes \Lambda^{1,0}(M,I)$ and

$$\mathrm{Id}_B \otimes J:\ B \otimes \Lambda^{1,0}(M,I) \longrightarrow B \otimes \Lambda^{0,1}(M,I)$$

be an endomorphism associated with $J \in \mathbb{H}$. We extend $\overline{\partial}$, $\overline{\partial}_J$ to operators

$$\overline{\partial}, \overline{\partial}_J:\ \Lambda^{0,p}(M,I) \otimes B \longrightarrow \Lambda^{0,p+1}(M,I) \otimes B.$$

using the Leibniz rule.

Proposition 3.3: In these assumptions. $\overline{\partial}^2 = \overline{\partial}_J^2 = 0$, and the anticommutator $\{\overline{\partial}, \overline{\partial}_J\}$ acts on $\Lambda^{0,*}(M,I)$ as a multiplication by an $\mathrm{End}(B)$-valued 2-form $\Theta_+ \in \Lambda^{0,2}(M,I) \otimes \mathrm{End}\, B$. Moreover, under the identification

$$\Lambda^{0,2}(M,I) \otimes \mathrm{End}\, B \cong \Lambda^{1,1}_{+,I}(M) \otimes \mathrm{End}\, B$$

(Proposition 2.6), Θ_+ corresponds to the $\Lambda^2_+(M)$-part of the curvature of B.

Proof: Let

$$\nabla_+:\ \Lambda^p_+(M) \otimes B \longrightarrow \Lambda^{p+1}_+(M) \otimes B$$

be the connection operator restricted to $\Lambda^*_+(M) \otimes B$, and $\nabla_+ = \nabla_+^{1,0} + \partial_+$ its Hodge decomposition. Clearly, ∇_+^2 is the $\Lambda^2_+(M)$-part of the curvature of B.

Now, Proposition 3.3 follows immediately from Theorem 2.10. Indeed, under the isomorphism (2.8), $x\overline{\partial}_J$ corresponds to $\nabla_+^{1,0}$; since the curvature of the Chern connection is of type $(1,1)$, we have $(\nabla_+^{1,0})^2 = 0$, hence $\overline{\partial}_J^2 = 0$. Similarly, the operator $\{x\overline{\partial}_J, y\overline{\partial}\}$ under the isomorphism (2.8) corresponds to $\{\nabla_+^{1,0}, \partial_+\} = \nabla_+^2$. ∎

3.3 Kodaira relations for $\overline{\partial}$, $\overline{\partial}_J$

Let (M, I, J, K) be a hyperkähler manifold. Consider the bicomplex

$$(\Lambda^{0,*}(M,I), \overline{\partial}, \overline{\partial}_J),$$

constructed in Subsection 2.6. Let

$$L_{\overline{\Omega}}:\ \Lambda^{0,p}(M,I) \longrightarrow \Lambda^{0,p+2}(M,I)$$

be an operator of exterrior multiplication by $\overline{\Omega}$ (Subsection 3.1), and

$$\overline{\partial}^*, \overline{\partial}_J^*:\ \Lambda^{0,p}(M,I) \longrightarrow \Lambda^{0,p-1}(M,I).$$

the operators Hermitian adjoint to $\overline{\partial}$, $\overline{\partial}_J$.

The following proposition is well known.

Proposition 3.4: In these assimptions, the following commutator relations hold.

$$[L_{\overline{\Omega}}, \overline{\partial}^*] = \overline{\partial}_J, \quad [L_{\overline{\Omega}}, \overline{\partial}_J^*] = -\overline{\partial}. \tag{3.2}$$

Proof: The proof of (3.2) is essentially the same as the proof of the usual Kodaira relations; see e.g. [V1], Proposition 4.2. ∎

The same argument, applied locally to $\operatorname{End}(B)$-valued forms, gives the following theorem.

Theorem 3.5: Let (M,I,J,K) be a hyperkähler manifold, B a holomorphic Hermitian vector bundle on (M,I),

$$\overline{\partial}, \overline{\partial}_J:\ \Lambda^{0,p}(M,I)\otimes B \longrightarrow \Lambda^{0,p+1}(M,I)\otimes B.$$

the operators constructed in Subsection 3.2, and $\overline{\partial}^*, \overline{\partial}_J^*$ the Hermitian adjoint operators. Then

$$[L_{\overline{\Omega}}, \overline{\partial}^*] = \overline{\partial}_J, \quad [L_{\overline{\Omega}}, \overline{\partial}_J^*] = -\overline{\partial}. \tag{3.3}$$

Proof: See [V1]. ∎

3.4 Kodaira-Nakano identities

The following theorem is the qD-analogue of the usual Kodaira-Nakano identity (or, rather, the identity used in the proof of Kodaira-Nakano vanishing)

Theorem 3.6: Let (M,I,J,K) be a hyperkähler manifold, B a holomorphic Hermitian vector bundle on (M,I),

$$\overline{\partial}, \overline{\partial}_J:\ \Lambda^{0,p}(M,I)\otimes B \longrightarrow \Lambda^{0,p+1}(M,I)\otimes B.$$

the operators constructed in Subsection 3.2, and $\overline{\partial}^*, \overline{\partial}_J^*$ the Hermitian adjoint operators. Consider the Laplacians

$$\Delta_{\overline{\partial}} := \{\overline{\partial}, \overline{\partial}^*\}, \quad \Delta_{\overline{\partial}_J} := \{\overline{\partial}_J, \overline{\partial}_J^*\}$$

(here, as elsewhere, $\{\cdot,\cdot\}$ denotes the anticommutator). Then

$$\Delta_{\overline{\partial}} - \Delta_{\overline{\partial}_J} = [\Theta_+, \Lambda_{\overline{\Omega}}], \tag{3.4}$$

where

$$\Theta_+ : \ \Lambda^{0,p}(M,I) \otimes B \longrightarrow \Lambda^{0,p+2}(M,I)$$

is an operator defined as

$$\Theta_+ := \{\overline{\partial}, \overline{\partial}_J\}$$

and identified with the $\Lambda^2_+(M) \otimes \operatorname{End} B$-part of the curvature of B as in Proposition 3.3.

Proof: Using the graded Jacobi identity and Theorem 3.5, we obtain

$$[\Theta_+, \Lambda_{\overline{\Omega}}] = -[\Lambda_{\overline{\Omega}}, \{\overline{\partial}, \overline{\partial}_J\}] = \{\overline{\partial}, \overline{\partial}^*\} - \{\overline{\partial}_J, \overline{\partial}_J^*\} = \Delta_{\overline{\partial}} - \Delta_{\overline{\partial}_J}.$$

■

4 Cohomology of hyperkähler manifolds

For the convenience of the reader, we recall here some well-known facts about the structure of $H^2(M)$ for M a compact, irreducible hyperkähler manifold; see [Bo2], [Bes], [Bea] and [F] for details.

4.1 $SU(2)$-action on $H^2(M)$

Let (M, I, J, K, g) be a compact, irreducible hyperkähler manifold. Since g is Kähler with respect to (I, J, K), we have

$$\nabla I = \nabla J = \nabla K = 0,$$

where ∇ denotes the Levi-Civita connection. Chern has shown that covariantly constant endomorphisms of $\Lambda^*(M)$ commute with the Laplacian (see [Bes]). Then the $SU(2)$-action generated by $I, J, K \in \mathbb{H}^*$ also commutes with the Laplacian. This gives an $SU(2)$-action on the space of harmonic forms on M. Identifying the harmonic forms with cohomology, we obtain an $SU(2)$-action on the cohomology as well.

Let $H^2(M) = H^2_+(M) \oplus H^2(M)_{SU(2)-inv}$ be a decomposition of $H^2(M)$ onto its weight 2 and weight 0 components. Using the weights of the Cartan algebra action as in the proof of Proposition 2.6, we find that

$$\dim H^{2,0}(M,I) = \dim H^{1,1}_+(M,I) = \dim H^{0,2}(M,I).$$

Since M is irreducible, $\dim H^{2,0}(M,I) = 1$ and the space $H^{1,1}_+(M,I)$ is one-dimensional. Let $H^2(M)_{SU(2)-inv}$ be the space of $SU(2)$-invariant classes. It is easy to check that $SU(2)$-invariant classes are all of type $(1,1)$ (e.g. [V1]).

Since $H^{1,1}_+(M,I)$ is one-dimensional and generated by the Kähler form ω_I, we have a decomposition

$$H^{1,1}(M,I) = \mathbb{C}\omega_I \oplus H^2(M)_{SU(2)-inv}. \tag{4.1}$$

Using the $\mathfrak{so}(1,4)$-action generated by the three Lefschetz $\mathfrak{sl}(2)$-triples associated with the Kähler structures I, J, K as in [V0], we can easily show that an $SU(2)$-invariant 2-form is primitive[1] (see e.g. [V1]).

This gives the following well-known statement ([V1]).

Claim 4.1: Let (M,I,J,K) be a compact, irreducible hyperkähler manifold. Then the space $H^{1,1}_{prim}(M,I)$ of primitive classes in $H^{1,1}(M,I)$ coincides with the space $H^2(M)_{SU(2)-inv}$ of $SU(2)$-invariant classes.

Proof: Since all $SU(2)$-invariant classes are primitive, $H^{1,1}_{prim}(M,I)$ contains $H^2(M)_{SU(2)-inv}$. Comparing the decomposition (4.1) with

$$H^{1,1}(M,I) = H^{1,1}_{prim}(M,I) \oplus \mathbb{C}\omega_I,$$

we find that $\dim H^{1,1}_{prim}(M,I) = \dim H^2(M)_{SU(2)-inv}$. ■

4.2 The Bogomolov-Beauville-Fujiki form

Let (M,I,J,K) be a compact hyperkähler manifold, and $\Omega := \omega_J + \sqrt{-1}\omega_K$ the holomorphic symplectic form on (M,I). F. Bogomolov ([Bo2]) defined the following bilinear symmetric 2-form on $H^{1,1}(M,I)$:

$$\widetilde{q}(\eta,\eta') := \int_M \eta \wedge \eta' \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1}, \tag{4.2}$$

[1] Recall that the **primitive classes** ([GH]) are cohomology classes which satisfy $\Lambda(\eta) = 0$, where $\Lambda : H^i(M) \longrightarrow H^{i-2}(M)$ is the dual Lefschetz operator. A $(1,1)$-class is primitive if and only if it is orthogonal to the Kähler form with respect to the Riemann-Hodge pairing.

where $n = \dim \mathbb{H} M$. Since $\Omega \wedge \overline{\Omega}$ is a positive (2,2)-form, $\widetilde{q}$ is positive on the Kähler cone of (M, I):

$$\forall \omega \in \mathcal{K} \quad \widetilde{q}(\omega, \omega) > 0 \tag{4.3}$$

An elementary linear-algebraic calculation similar to the proof of Riemann-Hodge bilinear relations implies that $\widetilde{q}(\eta, \eta) < 0$ for η primitive. Therefore, $\widetilde{q}$ has signature $(+, -, -, -, \dots)$ on $H^{1,1}(M, I) \cap H^2(M, \mathbb{R})$.

The form $\widetilde{q}$ is topological by its nature.

Theorem 4.2: ([F]) Let (M, I, J, K) be a compact, irreducible hyperkähler manifold of real dimension $4n$. Then there exists a bilinear, symmetric non-degenerate 2-form $q : H^2(M, \mathbb{Q}) \otimes H^2(M, \mathbb{Q}) \longrightarrow \mathbb{Q}$ such that

$$\int_M \eta^{2n} = q(\eta, \eta)^n, \tag{4.4}$$

for all $\eta \in H^2(M)$. Moreover, q is proportional to the form (4.2) on $H^{1,1}(M)$, and has signature $(+, +, +, -, -, -, ...)$.

■

Remark 4.3: If n is odd, the equation (4.4) determines q uniquely, otherwise – up to a sign. To choose a sign, we use (4.3)

Definition 4.4: Let (M, I, J, K) be a compact, irreducible hyperkähler manifold. A **Beauville-Bogomolov-Fujiki form** on M is a form $q : H^2(M, \mathbb{Q}) \otimes H^2(M, \mathbb{Q}) \longrightarrow \mathbb{Q}$ which satisfies (4.4), and take positive values on the Kähler cone of (M, I). Such a form always exists and is unique, by Theorem 4.2.

Remark 4.5: The Beauville-Bogomolov-Fujiki form is integer, but not unimodular on $H^2(M, \mathbb{Z})$.

The Beauville-Bogomolov-Fujiki form can be expressed in terms of the $SU(2)$-action on cohomology (Subsection 4.1) as follows:

Claim 4.6: Let (M, I, J, K) be a compact, irreducible hyperkähler manifold, and $(\cdot, \cdot)_{\mathcal{H}}$ the positive definite pairing on cohomology associated with the Euclidean metric on the space of harmonic forms induced by the Riemannian structure. Consider the form q' which is equal to $(\cdot, \cdot)_{\mathcal{H}}$ on the

3-dimensional space generated by ω_I, ω_J, ω_K, and to $-(\cdot,\cdot)_{\mathcal{H}}$ on its orthogonal complement. Then q' is proportional to the Beauville-Bogomolov-Fujiki form.

Proof: See e.g. [V3], Theorem 2.1. ■

This immediately gives the following corollary.

Corollary 4.7: Consider the natural $SU(2)$-action on the cohomology of a hyperkähler manifold. Then the Beauville-Bogomolov-Fujiki form is $SU(2)$-invariant.

■

Using the Hodge-Riemann bilinear relations, we can express $(\cdot,\cdot)_{\mathcal{H}}$ in terms of the product structure on cohomology. Together with Claim 4.6, this gives

$$q'(\eta_1,\eta_2) = \int_X \omega_I^{2n-2} \wedge \eta_1 \wedge \eta_2 - \frac{2n-2}{(2n-1)^2} \cdot \frac{\int_X \omega_I^{2n-1}\eta_1 \cdot \int_X \omega_I^{2n-1}\eta_2}{\int_X \omega_I^n} \quad (4.5)$$

for any $\eta_1, \eta_2 \in H^2(M)$ (see [V2], Claim 5.1). This formula is due to A. Beauville.

The following claim follows directly from (4.5) and Claim 4.1.

Claim 4.8: Let (M,I,J,K) be a irreducible, compact hyperkähler manifold, and $\eta \in H^{1,1}(M,I)$ a $(1,1)$-class. Then the following assertions are equivalent.

(i) $q(\eta,\omega_I) = 0$, where q is Beauville-Bogomolov-Fujiki form, and ω_I the Kähler class of (M,I)

(ii) η is primitive

(iii) η is $SU(2)$-invariant.

Proof: The equivalence of (ii) and (iii) is implied by Claim 4.1, and the equivalence of (i) and (iii) by (4.5). ■

5 The vanishing of cohomology

5.1 Cohomology vanishing for line bundles with $q(c_1(L), \omega) > 0$

The following result is implied immediately by the quaternionic Kodaira-Nakano identity (Theorem 3.6), in the same fashion as the usual Kodaira-Nakano vanishing follows from the the usual Kodaira-Nakano identity.

Proposition 5.1: Let (M, I, J, K) be a compact, irreducible hyperkähler manifold, $\dim_{\mathbb{R}} M = 4n$, and L a holomorphic line bundle on (M, I), such that $q(c_1(L), \omega_I) > 0$, where ω_I is the Kähler class of (M, I). Then the holomorphic cohomology $H^i(M, L)$ are zero for $i > n$.

Proof: Let η be a harmonic form representing $c_1(L)$. We may chose the Hermitian structure on L in such a way that η is equal to the curvature of L (see [GH]). Let ω denote the Kähler form of (M, I). Abusing the notation, we denote the Kähler class of (M, I) by the same letter.

The cohomology class

$$\kappa := c_1(L) - \frac{q(c_1(L), \omega)}{q(\omega, \omega)} \omega$$

clearly satisfies $q(\kappa, \omega) = 0$. Therefore, κ is $SU(2)$-invariant (Claim 4.8). Since η is harmonic, the harmonic form

$$\eta - \frac{q(c_1(L), \omega)}{q(\omega, \omega)} \omega \tag{5.1}$$

representing κ is also $SU(2)$-invariant. Let $\widetilde{\omega}$ be the form

$$\omega \in \Lambda^{1,1}_{+,I}(M)$$

considered as an element in $\Lambda^{0,2}(M, I)$ using the isomorphism, constructed Proposition 2.6. By Proposition 3.3,

$$\Theta_+ = \{\overline{\partial}, \overline{\partial}_J\} = \frac{q(c_1(L), \omega)}{q(\omega, \omega)} \widetilde{\omega}. \tag{5.2}$$

Clearly, $\omega_I, \omega_J, \omega_K$ form a 3-dimensional irreducible $SU(2)$-invariant subspace of $\Lambda^2(M)$. A trivial calculation is used to show that $\widetilde{\omega}$ is in fact equal to $\overline{\Omega}$. This gives

$$\Theta_+ = \{\overline{\partial}, \overline{\partial}_J\} = \lambda L_{\overline{\Omega}}, \tag{5.3}$$

where $\lambda = \frac{q(c_1(L),\omega)}{q(\omega,\omega)}$ is a positive constant. Comparing (5.3), Kodaira-Nakano identity (3.4) and the quaternionic Lefschetz theorem (Proposition 3.1), we obtain

$$\Delta_{\overline{\partial}} - \Delta_{\overline{\partial}_J} = [\Theta_+, \Lambda_{\overline{\Omega}}] = \lambda H_{\overline{\Omega}}. \tag{5.4}$$

On $(0,i)$-forms this operator acts as $(i-n)\lambda$. Given a harmonic form

$$\nu \in \ker \Delta_{\overline{\partial}} \subset \Lambda^{0,i}(M,I) \otimes L,$$

we can obtain

$$0 = \Delta_{\overline{\partial}}(\nu) = \Delta_{\overline{\partial}_J}(\nu) + (i-n)\lambda\nu. \tag{5.5}$$

Since

$$(\Delta_{\overline{\partial}_J}(\nu), \nu) = (\overline{\partial}_J\eta, \overline{\partial}_J\eta) + (\overline{\partial}^*_J\eta, \overline{\partial}^*_J\eta) \geqslant 0, \tag{5.6}$$

(5.5) leads to $(\nu,\nu) = 0$ for $i > n$. The harmonic $(0,i)$-forms are identified with the i-th holomorphic cohomology of L as usual. We proved Proposition 5.1. ■

Remark 5.2: Let W be a Hermitian vector space. A **positive operator** $A: W \longrightarrow W$ is an operator which satisfies $(A(x), x) \geqslant 0$ for all $x \in W$. A is **positive definite** if this inequality is strict for all non-zero x. From (5.6), we obtain that the Laplacians $\Delta_{\overline{\partial}_J}$ and $\Delta_{\overline{\partial}}$ are positive. If

$$\Delta_{\overline{\partial}} = \Delta_{\overline{\partial}_J} + A,$$

where A is positive definite, then $\ker \Delta_{\overline{\partial}} = 0$. This argument is used quite often in geometry and analysis.

Remark 5.3: Let (M,I,J,K) be a compact, irreducible hyperkähler manifold, $\dim_{\mathbb{H}} M = n$, and L a holomorphic line bundle on (M,I). Then Serre's duality gives $H^i(M,L)^* \cong H^{n-i}(M,L^*)$, because the canonical class of M is trivial. Therefore, Proposition 5.1 implies that $H^i(M,L)$ vanish for all $i < n$ if L is a holomorphic line bundle on (M,I) with $q(c_1(L),\omega) < 0$.

5.2 The dual Kähler cone and vanishing

Let (M,I) be a Kähler manifold.

Definition 5.4: The **Kähler cone** $\mathcal{K} \subset H^{1,1}(M,I)$ for (M,I) is the set of all Kähler classes $\omega \in H^{1,1}_I(M,\mathbb{R})$, where $H^{1,1}_I(M,\mathbb{R})$ denotes the intersection $H^{1,1}(M,I) \cap H^2(M,\mathbb{R})$. Clearly, $\mathcal{K}$ is a convex cone in $H^{1,1}_I(M,\mathbb{R})$.

Now, let (M, I, J, K) be a hyperkähler manifold, and $\mathcal{K} \subset H^{1,1}_I(M, \mathbb{R})$ the Kähler cone of (M, I), and $q : H^{1,1}_I(M, \mathbb{R}) \times H^{1,1}_I(M, \mathbb{R}) \longrightarrow \mathbb{R}$ the Beauville-Bogomolov-Fujiki form. We define **the dual Kähler cone**

$$\mathcal{K}\check{} \subset H^{1,1}_I(M, \mathbb{R})$$

as

$$\mathcal{K}\check{} := \{x \in H^{1,1}_I(M, \mathbb{R}) \ \mid \ \forall y \in \mathcal{K}, \ \ q(x, y) > 0\}$$

It is an open, convex cone. Since a product of two Kähler forms is positive, we have $\mathcal{K}\check{} \supset \mathcal{K}$.

Denote by $\overline{\mathcal{K}}\check{}$ the closure of $\mathcal{K}\check{}$ in $H^{1,1}_I(M, \mathbb{R})$, and by $-\overline{\mathcal{K}}\check{}$ the opposite cone. Clearly,

$$\overline{\mathcal{K}}\check{} := \{x \in H^{1,1}_I(M, \mathbb{R}) \ \mid \ \forall y \in \mathcal{K}, \ \ q(x, y) \geqslant 0\}$$

and

$$-\overline{\mathcal{K}}\check{} := \{x \in H^{1,1}_I(M, \mathbb{R}) \ \mid \ \forall y \in \mathcal{K}, \ \ q(x, y) \leqslant 0\}$$

Proposition 5.1 immediately leads to the following corollary.

Corollary 5.5: Let (M, I) be a compact, irreducible, holomorphically symplectic Kähler manifold, $\dim_{\mathbb{C}} M = 2n$, and L a holomorphic line bundle on (M, I) with $c_1(L) \notin \overline{\mathcal{K}}\check{}$. Then the holomorphic cohomology $H^i(M, L)$ are zero for all $i < n$.

■

Now we can prove the main result of this paper.

Theorem 5.6: Let (M, I, J, K) be a compact, irreducible hyperkähler manifold, and L a holomorphic line bundle on (M, I) with $c_1(L) \neq 0$. Then one of the following holds.

(i) $c_1(L) \in \overline{\mathcal{K}}\check{}$; then $H^i(L) = 0$ for $i > \frac{\dim_{\mathbb{C}} M}{2}$.

(ii) $c_1(L) \in -\overline{\mathcal{K}}\check{}$; then $H^i(L) = 0$ for $i < \frac{\dim_{\mathbb{C}} M}{2}$.

(iii) $c_1(L)$ does not lie in $-\overline{\mathcal{K}}\check{} \cup \overline{\mathcal{K}}\check{}$; then $H^i(L) = 0$ for $i \neq \frac{\dim_{\mathbb{C}} M}{2}$.

Proof: Denote $\frac{\dim_{\mathbb{C}} M}{2}$ by n. Theorem 5.6 (iii) is a direct consequence of Corollary 5.5. Indeed, in this case

$$H^i(L) = 0 \text{ for } \quad i < n$$

and

$$H^i(L^*) = 0 \text{ for } \quad i < n, \tag{5.7}$$

because the Chern classes of both L and L^* do not lie in $\overline{\mathcal{K}}\check{}$. However, by Serre's duality, (5.7) is equivalent to

$$H^i(L) = 0 \text{ for } \quad i > n.$$

Let us prove Theorem 5.6 (i). Since $c_1(L) \in \overline{\mathcal{K}}\check{}$, we may assume that $q(c_1(L), \omega) \geqslant 0$ for all $\omega \in \mathcal{K}$. Unless $q(c_1(L), \omega) = 0$ for all Kähler classes ω, the assertion of Theorem 5.6 (i) is obtained from Proposition 5.1. However, if $q(c_1(L), \omega) = 0$ for all Kähler classes, $c_1(L) = 0$, because q is non-degenerate and the Kähler classes generate $H^{1,1}_I(M, \mathbb{R})$. Theorem 5.6 (i) is obtained from (ii) by Serre's duality. ∎

The classes $\eta \notin -\overline{\mathcal{K}}\check{} \cup \overline{\mathcal{K}}\check{}$ can be also characterized as follows.

Claim 5.7: Let (M, I, J, K) be a compact, irreducible hyperkähler manifold, and $\eta \in H^{1,1}_I(M, \mathbb{R})$ a non-zero cohomology class. Then the following conditions are clearly equivalent.

(i) $\eta \notin -\overline{\mathcal{K}}\check{} \cup \overline{\mathcal{K}}\check{}$

(ii) $q(\eta, \omega_1) > 0$ and $q(\eta, \omega_2) < 0$ for some Kähler forms ω_1, ω_2 on (M, I)

(iii) η is primitive with respect to some Kähler form on (M, I); or, equivalently, $q(\eta, \omega) = 0$ (see Claim 4.8).

(iv) The class η is $SU(2)$-invariant with respect to some hyperkähler structure (I, J', K') on M.

Proof: The equivalence of (i) and (ii) is clear. The equivalence of (iii) and (iv) is implied by Claim 4.8. The implication (ii) $\Rightarrow$ (iii) is clear, because the Kähler cone is connected, hence from $q(\eta, \omega_1) > 0$ and $q(\eta, \omega_2) < 0$ it follows that $q(\eta, \omega_3) = 0$ for some Kähler form. Finally, (iii) $\Rightarrow$ (ii) is obtained as follows: given a Kähler class ω, with $q(\eta, \omega) = 0$, take a neighbourhood U of ω in the Kähler cone. The function $U \xrightarrow{v} \mathbb{R}$, $v(\omega') = q(\eta, \omega')$ is non-zero and linear, hence it takes positive and negative values in any open neighbourhood of ω. ∎

5.3 Cohomology vanishing for vector bundles of arbitrary rank

A version of Theorem 5.6 can be stated for holomorphic bundles of arbitrary rank, as follows.

Theorem 5.8: Let (M, I, J, K) be a compact, irreducible hyperkähler manifold, L a holomorphic line bundle on (M, I) with $c_1(L) \neq 0$, and B an arbitrary holomorphic vector bundle on (M, I). Then there exists a sufficiently big number N_0, such that for any integer $N > N_0$ one of the following holds.

(i) $c_1(L) \in \overline{\mathcal{K}}\check{}$; then $H^i(L^N \otimes B) = 0$ for $i > \frac{\dim_{\mathbb{C}} M}{2}$.

(ii) $c_1(L) \in -\overline{\mathcal{K}}\check{}$; then $H^i(L^N \otimes B) = 0$ for $i < \frac{\dim_{\mathbb{C}} M}{2}$.

(iii) $c_1(L)$ does not lie in $-\overline{\mathcal{K}}\check{} \cup \overline{\mathcal{K}}\check{}$; then $H^i(L^N \otimes B) = 0$ for $i \neq \frac{\dim_{\mathbb{C}} M}{2}$.

Proof: The proof of Theorem 5.8 is similar to the Kodaira-Nakano vanishing for vector bundles of arbitrary rank. The same argument as used in the proof of Theorem 5.6 can be employed to deduce Theorem 5.8 from the following statement.

Proposition 5.9: Let (M, I, J, K) be a compact, irreducible hyperkähler manifold, L a holomorphic line bundle on (M, I) with $c_1(L) \neq 0$, and B an arbitrary holomorphic vector bundle on (M, I). Assume that $q(c_1(L), \omega) > 0$, where ω is the Kähler form of (M, I). Then there exists a sufficiently big number N_0, such that for any integer $N > N_0$ $H^i(L^N \otimes B) = 0$ for all $i > \frac{\dim_{\mathbb{C}} M}{2}$.

Proof: To prove Proposition 5.9, we use the formula (3.4) again:

$$\Delta_{\overline{\partial}} - \Delta_{\overline{\partial}_J} = -[\Theta_+, \Lambda_{\overline{\Omega}}], \tag{5.8}$$

where $\Delta_{\overline{\partial}}$, $\Delta_{\overline{\partial}_J}$ are the Laplacians on $L^N \otimes B$, and Θ_+ is the $\Lambda^{1,1}_{+,I}(M) \otimes \operatorname{End}(L^N \otimes B)$-part of the curvature of $L^N \otimes B$, considered as an operator on $\Lambda^{(0,*)}(M) \otimes (L^N \otimes B)$ as in the proof of the quaternionic Dolbeault Kodaira-Nakano identity (Theorem 3.6). Since the curvature is additive on tensor product, we have

$$\Theta_+ = \Theta_B + N\Theta_L,$$

where Θ_B, Θ_L are $\Lambda^2_+(M)$-parts of the curvatures of B and L. The same argument as used in the proof of (5.2) implies that $\Theta_L = \lambda\overline{\Omega}$, where $\lambda = \frac{q(c_1(L),\omega)}{q(\omega,\omega)}$. Then, as the Lefschetz formula (Proposition 3.1) implies,

$$-[\Theta_+, \Lambda_{\overline{\Omega}}] = -[\Theta_B, \Lambda_{\overline{\Omega}}] + V,$$

where V is a scalar operator acting on $(0,i)$-forms as $\lambda(i-n)N$, $n = \frac{\dim_C M}{2}$. Clearly, $-[\Theta_B, \Lambda_{\overline{\Omega}}] + V$ is positive definite for N sufficiently big and $i > n$. From Remark 5.2 we obtain immediately that $\ker \Delta_{\overline{\partial}} = 0$ whenever $-[\Theta_B, \Lambda_{\overline{\Omega}}] + V$ is positive definite. This proves Proposition 5.9. Theorem 5.8 is proven. ■

6 Vanishing of cohomology and nef-classes with $q(\eta, \eta) = 0$

6.1 Nef classes

The following immensely important theorem was proven by J.-P. Demailly and M. Paun.

Theorem 6.1: ([DP]) Let M be a compact Kähler manifold, and $\mathcal{X}$ the set of all closed analytic subvarieties $X \subset M$ of positive dimension. Consider the set of all $(1,1)$-classes

$$\widetilde{\mathcal{K}} := \{\eta \in H^{1,1}(M)\mathcal{H}^{\in}(\mathcal{M}, \mathbb{R}) \ \mid\ \forall \mathcal{X} \in \mathcal{X},\ \int_{\mathcal{X}} \eta^{\dim \mathcal{X}} > /\}$$

Then the Kähler cone of M coincides with one of the connected components of $\widetilde{\mathcal{K}}$. ■

Remark 6.2: The converse assertion is trivial: if η is a Kähler class, then $\int_X \eta^i > 0$ for all analytic cycles $X \subset M$, $\dim X = i$.

Definition 6.3: Let M be a Kähler manifold, and $\eta \in H^{1,1}(M)$ a real $(1,1)$-class. Then η is called **nef** (numerically effective) if η belongs to a closure $\overline{\mathcal{K}}$ of the Kähler cone $\mathcal{K}$ of M. The closure $\overline{\mathcal{K}}$ is called **the nef cone**. A nef line bundle on M is a line bundle with $c_1(L)$ nef; a nef divisor D is one with nef cohomology class.

6.2 Nef classes on hyperkähler manifolds

Consider a compact, irreducible hyperkähler manifold (M, I, J, K). Let L be a holomorphic line bundle on (M, I) which is nef and satisfies

$$q(c_1(L), c_1(L)) = 0.$$

It was conjectured ([GHJ], [Saw]) that L is base point free, that is, defines a holomorphic map

$$(M, I) \longrightarrow \mathbb{P}H^0(L^N) \tag{6.1}$$

for N sufficiently big. If this is true, then (6.1) is a Lagrangian fibration onto its image ([M1]). A special case of this conjecture was recently proven by D. Matsushita ([M2]). This motivates our interest in the geometry of nef-classes satisfying $q(\eta, \eta) = 0$.

Proposition 6.4: Let (M, I, J, K) be a compact, irreducible hyperkähler manifold, $\eta \in H^{1,1}_I(M, \mathbb{R})$ a non-zero nef class on (M, I), satisfying $q(\eta, \eta) = 0$, and ω a rational Kähler class on (M, I). Then

(i) $q(\omega, \eta) > 0$

(ii) Choose a positive real number $\varepsilon < \frac{q(\eta,\omega)}{q(\omega,\omega)}$. Then $\eta - \varepsilon\omega$ lies outside of $\overline{\mathcal{K}}\check{} \cup -\overline{\mathcal{K}}\check{}$.

Proof: Proposition 6.4 (i) is clear. Indeed, if $q(\omega, \eta) = 0$, then $q(\eta, \eta) < 0$, because the form q has signature $(+, -, -, ...-)$ on $H^{1,1}(M, \mathbb{R})$. On the other hand, $q(\omega, \eta) \geqslant 0$, because η lies in the closure of the Kähler cone.

Let us prove Proposition 6.4 (ii). Since $\varepsilon < \frac{q(\eta,\omega)}{q(\omega,\omega)}$, the number

$$\lambda := q(\omega, \eta - \varepsilon\omega) \tag{6.2}$$

is positive. Therefore, $\eta - \varepsilon\omega \notin -\overline{\mathcal{K}}\check{}$. To prove Proposition 6.4 (ii), it remains to find a Kähler class ω' which satisfies $q(\omega', \eta - \varepsilon\omega) < 0$.

For any $\delta > 0$, $\eta + \delta\omega$ is a Kähler class, as follows from Theorem 6.1. Choose a positive number $\delta < \frac{\lambda\varepsilon}{q(\eta\omega)}$, where λ is the number defined in (6.2). Then

$$q(\eta + \delta\omega, \eta - \varepsilon\omega) = -\varepsilon\lambda + \delta q(\omega, \eta) = q(\omega, \eta)\left(\delta - \frac{\lambda\varepsilon}{q(\eta, \omega)}\right) < 0.$$

Proposition 6.4 (ii) is proven. ■

6.3 A vanishing theorem and its applications

From Proposition 6.4, the following theorem is apparent.

Theorem 6.5: Let (M, I, J, K) be a compact, irreducible hyperkähler manifold, $\dim_{\mathbb{H}} M = n$, and L a non-trivial holomorphic bundle on (M, I) which is nef and satisfies $q(c_1(L), c_1(L)) = 0$. Consider an ample line bundle H on (M, I). Then there exists N_0 such that for all integers $N > N_0$,

$$H^i(L^N \otimes H^*) = 0 \quad \text{for} \quad i \neq n. \tag{6.3}$$

Proof: Let $N_0 = \frac{1}{\varepsilon}$, where

$$\varepsilon = \frac{q(c_1(L), c_1(H))}{q(c_1(H), c_1(H))}.$$

Then $Nc_1(L) - c_1(H) \notin \overline{\mathcal{K}}\check{} \cup -\overline{\mathcal{K}}\check{}$ as follows from Proposition 6.4. The vanishing of (6.3) then follows from Theorem 5.6. ■

Theorem 6.5 has an immediate corollary.

Corollary 6.6: Let (M, I, J, K) be a compact, irreducible hyperkähler manifold, $\dim_{\mathbb{H}}(M) > 1$, L a non-trivial holomorphic bundle on (M, I) which is nef and satisfies $q(c_1(L), c_1(L)) = 0$, and D an ample divisor on (M, I). Then, for sufficiently big $N > N_0$, the natural restriction map

$$H^0(L^N) \longrightarrow H^0(L^N\Big|_D)$$

is surjective.

Proof: The following exact sequence is well known

$$0 \longrightarrow L^N(-D) \longrightarrow L^N \longrightarrow L^N\Big|_D \longrightarrow 0.$$

By Theorem 5.6, $H^1(L^N(-D)) = 0$. Then the long exact sequence of cohomology gives

$$0 \longrightarrow H^0(L^N(-D)) \longrightarrow H^0(L^N) \longrightarrow H^0\left(L^N\Big|_D\right) \longrightarrow 0$$

This proves Corollary 6.6. ■

Corollary 6.6 can be generalized as follows.

Theorem 6.7: Let (M, I, J, K) be a irreducible hyperkähler manifold, and $X \subset (M, I)$ a subvariety of dimension $\dim_{\mathbb{C}} X > \frac{1}{2} \dim_{\mathbb{C}} M$. Assume that X is a complete intersection of ample divisors. Consider a holomorphic line bundle L on (M, I) with $c_1(L)$ nef and $q(c_1(L), c_1(L)) = 0$. Then the natural restriction map is surjective on holomorphic sections:

$$H^0(L^N) \longrightarrow H^0\left(L^N\Big|_X\right) \longrightarrow 0,$$

for a sufficiently big power of L.

Proof: Let $X = \bigcup_{i=1}^k H_i$, where $k = \operatorname{codim} X$, and H_i are ample divisors. Consider the Koszul resolution of $L^N\Big|_X$,

$$\begin{aligned} 0 &\longrightarrow L^N(-H_1 - H_2 - \cdots - H_k) \longrightarrow \ldots \\ &\longrightarrow \bigoplus_{i>j} L^N(-H_i - H_j) \longrightarrow \bigoplus_i L^N(-H_i) \\ &\longrightarrow L^N \longrightarrow L^N\Big|_X \longrightarrow 0. \end{aligned} \tag{6.4}$$

By Theorem 6.5, the cohomology of the terms $L^N(-H_i - H_j - ...)$ of (6.4) vanish up to degree n. Therefore, the $E_0^{p,q}$-term of the associated spectral sequence looks as follows

$$\begin{array}{ccccc} H^n(L^N(-H_1 - \ldots - H_k)) & \ldots & H^n\left(\bigoplus_i L^N(-H_i)\right) & H^n(L^N) & H^n\left(L^N\big|_X\right) \\ 0 & \ldots & 0 & H^{n-1}(L^N) & H^{n-1}\left(L^N\big|_X\right) \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & H^0(L^N) & H^0\left(L^N\big|_X\right) \end{array} \tag{6.5}$$

From (6.5) it is clear that the only non-trivial differential of (6.5) mapping to $H^0\left(L^N\Big|_X\right)$ is

$$d_1: \; H^0(L^N) \longrightarrow H^0\left(L^N\Big|_X\right) \tag{6.6}$$

which is identified with the restriction map. Since the complex (6.4) is exact, the spectral sequence (6.5) converges to zero. Therefore, the differential (6.6) is surjective. This proves Theorem 6.7. ∎

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